Girraween High School

2021

TRIAL HIGHER SCHOOL CERTIFICATE

EXAMINATION

Mathematics Extension 2

Total Marks: 100

Section 1 (Pages 2 – 5) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

Section 2 (Pages 5-14) **90 marks**

- Attempt Q11 - Q16
- Allow about 2 hours and 45 minutes for this section

General Instructions

- Reading time: 10 minutes
- Working time: 3 Hours
- Write using a black or blue pen
- Board approved calculators may be used
- You may use the NESA Mathematics reference sheet
- Answer multiple-choice questions by writing your answer (A,B,C or D) on the answer paper provided by the school (e.g. (1) A(2) C etc.)
- Answer questions 11-16 on the answer paper provided by the school and show all relevant mathematical reasoning and/or calculations.
- At the end of the examination, scan and send your completed paper to the assignment AS A SINGLE PDF labelled 2021 Extension 2 Trial Paper Answers in your class's (12Ext2G or 12Ext2R) Google Classroom.

Section 1 (10 marks)

Attempt Questions 1-10

Allow about 15 minutes for this section

Question 1

The magnitude of the vector $3i - 2j + k$ is

(A) $\sqrt{\mathbf{14}}$ **(B)** $\mathbf{14}$ **(C)** $\sqrt{\mathbf{6}}$ **(D)** $\mathbf{6}$

Question 2

The Cartesian equation of the line $\underset{\sim}{r} = \binom{2}{1} + \lambda\binom{3}{-4}$ is

(A) $\boldsymbol{4x + 3y - 11 = 0}$ **(B)** $\boldsymbol{4x + 3y + 5 = 0}$ **(C)** $\boldsymbol{4x + 3y - 1 = 0}$

(D) $\boldsymbol{4x - 3y - 1 = 0}$

Question 3

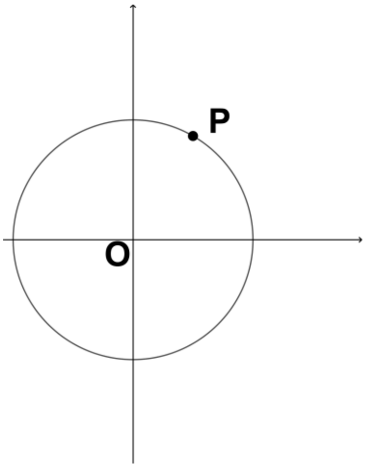
If one of the roots of $z^2 + pz + q = 0, p, q$ real is $z = 2 - 4i$ then

(A) $\boldsymbol{p = 4, q = -20}$ **(B)** $\boldsymbol{p = 2, q = -10}$ **(C)** $\boldsymbol{p = -4, q = 20}$ **(D)** $\boldsymbol{p = -2, q = 10}$

Multiple choice continues on the following page

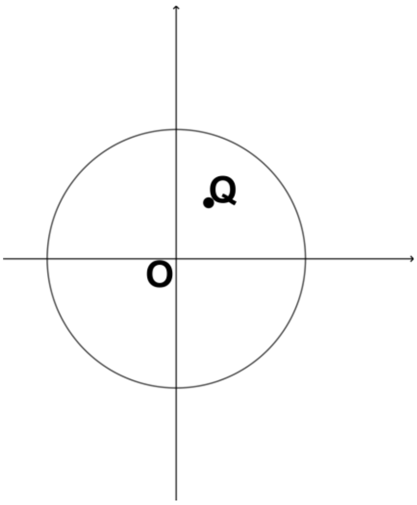
Question 4

In the diagram below, $\overrightarrow{OP} = z, |z| > 1$ and $\frac{\pi}{4} < Arg\ z < \frac{\pi}{2}$. *(see diagram)*

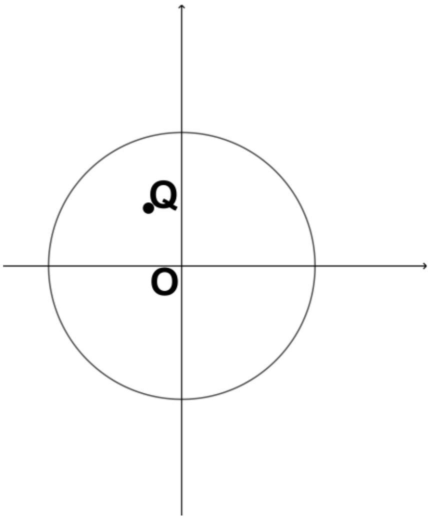


If $\overrightarrow{OQ} = z^2$ then Q would be at

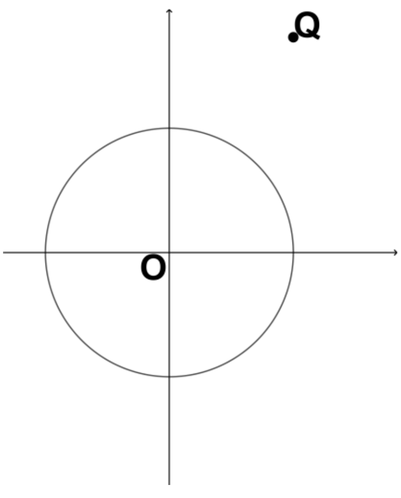
(A)



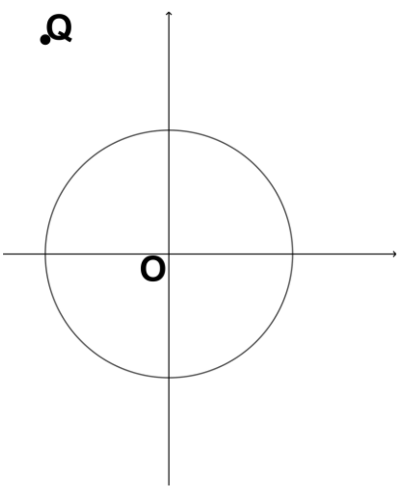
(B)



(C)



(D)



Multiple choice continues on the following page

Question 5

$2e^{\frac{3\pi i}{4}} =$

(A) $\sqrt{2} - i\sqrt{2}$ **(B)** $-\sqrt{2} + i\sqrt{2}$ **(C)** $-\sqrt{3} + i$ **(D)** $\sqrt{3} - i$

Question 6

A particle is moving in simple harmonic motion between $x = 1$ and $x = 5$ with period $\frac{\pi}{3}$ seconds. The equation for its position in terms of time could be

(A) $x = 2sin3t + 3$ **(B)** $x = 4sin3t + 3$ **(C)** $x = 4sin6t + 3$ **(D)** $x = 2sin6t + 3$

Question 7

A counterexample to the statement $3^n - 1$ is divisible by 4, $n \in Z^+$ is

(A) $3^2 - 1$ **(B)** $3^3 - 1$ **(C)** $3^4 - 1$ **(D)** $3^6 - 1$

Question 8

The negation of the statement "If n is a positive integer, $3^n - 1$ is divisible by 4" is

(A) "n is a positive integer so $3^n - 1$ is divisible by 4"

(B) "n is NOT a positive integer BUT $3^n - 1$ is divisible by 4"

(C) "n is a positive integer BUT $3^n - 1$ is NOT divisible by 4"

(D) "n is NOT a positive integer and $3^n - 1$ is NOT divisible by 4"

Multiple choice continues on the following page

Question 9

$$\int \frac{1}{x^2 - 4x + 7}\,.dx =$$

(A) $\boldsymbol{tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + C}$

(B) $\boldsymbol{ln\left(\frac{x-2+\sqrt{3}}{x+2+\sqrt{3}}\right) + C}$

(C) $\boldsymbol{\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + C}$

(D) $\boldsymbol{ln\left(\frac{x+2+\sqrt{3}}{x-2-\sqrt{3}}\right) + C}$

Question 10

$$\int x^{2021} lnx\,.dx =$$

(A) $\boldsymbol{\frac{x^{2022}lnx}{2022} + C}$

(B) $\boldsymbol{\frac{x^{2022}}{2022}\left(lnx - \frac{1}{2022}\right) + C}$

(C) $\boldsymbol{\frac{x^{2022}}{2022}(lnx - 1) + C}$

(D) $\boldsymbol{\frac{x^{2022}}{2022}(lnx - \frac{1}{2021}) + C}$

Section II (90 marks)

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section

Start the answers to each question on a separate page in your answer booklet.

In Questions 11-16 your responses should include all relevant mathematical reasoning and/ or calculations.

Question 11 ***(15 marks)*** **Marks**

(a) If $z = 1 + 2i$ and $w = 3 + i$

(i) Find $\frac{z}{w}$ in Cartesian form **2**

(ii) HENCE show that $tan^{-1}(2) - tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$ **3**

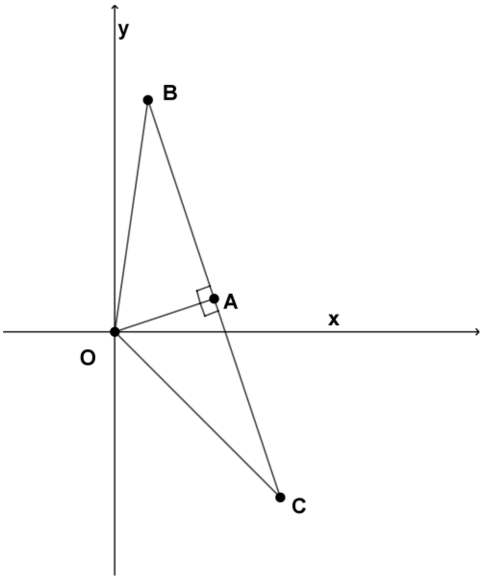
Question 11 continues on the following page

Question 11 (continued) **Marks**

(b) Use DeMoivre's Theorem to find $\left(\sqrt{3} - i\right)^5$. Give your answer in Cartesian form. **3**

(c) In the diagram below, $\overrightarrow{OA} = z_1$, $\overrightarrow{OB} = z_2$ and $\overrightarrow{OC} = z_3$.

$\angle OAB = \angle OAC = 90^o$ and $AB = AC = \sqrt{3}\,OA$.



(i) Show that $z_2 = (1 + i\sqrt{3}\,)z_1$ **1**

(ii) Show that $z_2 + z_3 = 2z_1$ **1**

(d) Sketch on an Argand diagram: **2**

$|z - i| \leq 5 \; and \; \frac{\pi}{4} < Arg(z) \leq \frac{\pi}{2}$

(e) (i) Show that $e^{ni\theta} - e^{-ni\theta} = 2i sin\, n\theta$ **1**

(ii) Hence show that $16 sin^4\theta = 2cos\, 4\theta - 8cos\, 2\theta + 6$ **1**

(You may assume $e^{ni\theta} + e^{-ni\theta} = 2cos\, n\theta$)

(iii) HENCE find $\int sin^4\theta . d\theta$ **1**

Examination continues on the following page

Question 12 ***(15 marks)*** **Marks**

(a) (i) Express $\frac{9x-9}{(x-2)^2(x+1)}$ in the form $\frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{C}{x+1}$ **3**

(ii) Hence find $\int \frac{9x-9}{(x-2)^2(x+1)}\,.dx$ **1**

(b) Using the substitution $t = tan\left(\frac{x}{2}\right)$ or otherwise, find **3**

$$\int \frac{1}{sinx + cosx + 1}.dx$$

(c) Use integration by parts to find $\int_0^{\pi} x\ sinx.dx$ **2**

(d) (i) Prove $\int_0^{a} f(a-x).dx\ = \int_0^{a} f(x).dx$ **1**

(ii) HENCE find $\int_0^{\pi} x\ sinx.dx$ **2**

(e) (i) If $I_n = \int sin^n\,x.dx$, show that $I_n = -\frac{cos\ x\ sin^{n-1}x}{n} + \frac{n-1}{n}I_{n-2}$ **2**

(ii) HENCE find $\int sin^4x.dx$ **1**

Examination continues on the following page

Question 13 ***(15 Marks)*** **Marks**

(a) If a is even, prove that $a^2 + 2a$ is always divisible by 8. **2**

(b) Prove by contradiction that $\sqrt{5}$ is irrational. **3**

(c) Prove by contraposition that if $n^2 - 2n$ is odd, n is odd. **2**

(d) Prove by induction that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \geq \frac{3}{2} - \frac{1}{n+1}$ **4**

$\forall\, n \geq 1, n \in Z^+$

(e)(i) Prove $a^2 + b^2 \geq 2ab \;\; \forall a, b \in R$ **1**

(ii) Hence prove $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4, a, b > 0$ **2**

(iii) Hence prove $cosec^2\theta + sec^2\theta \geq 4 \;\; \forall\, \theta$ **2**

Examination continues on the following page

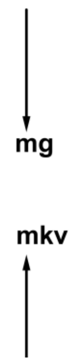
Question 14 ***(15 marks)*** **Marks**

(a) A particle moves in Simple Harmonic Motion (SHM) about $x = 0$ with acceleration given by $\ddot{x} = -9x$. If the velocity of the particle is $9m/s$ when $x = 4$,

(i)Express the particle's motion in the form $v^2 = n^2(a^2 - x^2)$ and state the period and amplitude of the motion. **3**

(ii) Find the particle's maximum velocity. **1**

(b) A particle with mass m is dropped from a stationary balloon. If the force of gravity on the particle is mg (downwards obviously!) and the particle experiences air resistance proportional to its speed (mkv) in the opposite direction to its motion *(see diagram)*



(i) Find an expression for the particle's acceleration in terms of velocity and find the velocity it can't exceed as it falls (terminal velocity) if $g = 10m/s^2$ and $k = \frac{1}{6}$ *(Note: Down is positive in this question!!!)* **2**

(ii) Show $\boldsymbol{t = -6ln\left(1 - \frac{v}{60}\right)}$ and find the time at which it hits the ground if it hits at half the terminal velocity. **2**

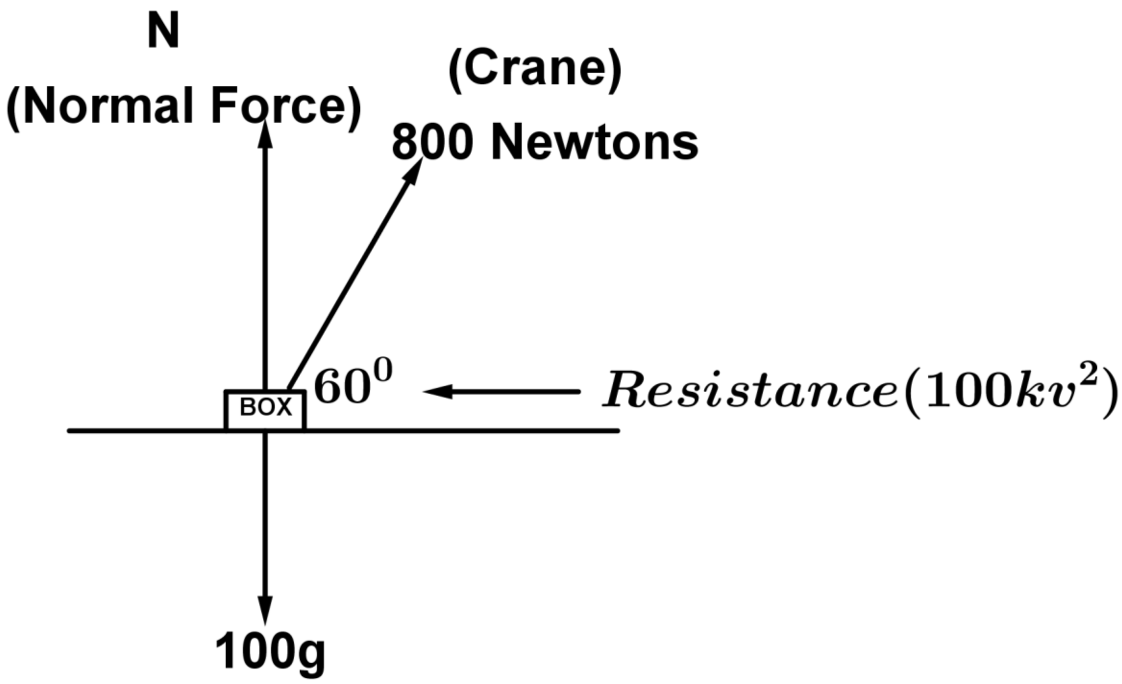
(iii) Show that $x = 60t + 360(e^{-\frac{t}{6}} - 1)$ and find the height from which the particle was dropped. **2**

Question 14 continues on the following page

Question 14 (continued) **Marks**

(c) A box with a mass of $100kg$ is attached to a crane by a taut rope which is at an angle of 60^o to the horizontal. The box is initially stationary but then starts to move horizontally along the ground as the crane pulls it with an overall force of 800 Newtons.

Once the box starts moving it experiences resistance in the form of friction of $100kv^2$ where v is its velocity *(see diagram)*



(i) Taking $g = 10m/s^2$, by resolving forces vertically and assuming the box stays on the ground, find the magnitude of the normal force. **1**

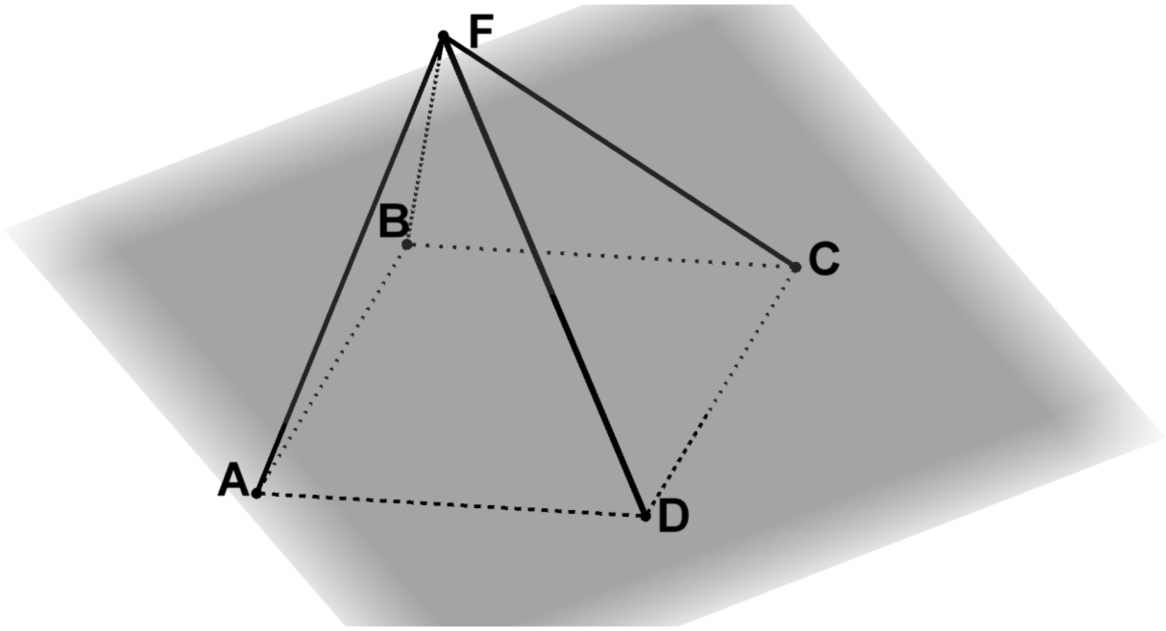
(ii) By resolving forces horizontally, show that $\ddot{x} = 4 - kv^2$ and find the value of k if the box has a limiting horizontal velocity of $20m/s$. **2**

(iii) Show that $x = -\frac{1}{2k}ln\left(1 - \frac{kv^2}{4}\right)$ and find how far the box has moved when it is moving at $10m/s$. **2**

Examination continues on the following page

Question 15 ***(15 marks)*** **Marks**

$\boldsymbol{A(1,3,-2)}$ $\boldsymbol{B(7,11,22)}$ $\boldsymbol{C(31,17,14)}$ and $D(25,9,-10)$ form a quadrilateral on a plane in 3 dimensional space. F $(24,-14,12)$ is another point in 3 dimensional space which is NOT on this plane. *(see diagram)*



(a) Show that A lies on the plane $4x-12y+3z+38=0$ **1**

(b) Show that the equation of the line AC is $\begin{pmatrix}1\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}30\\14\\16\end{pmatrix}$ **1**

(c) The equation of the line BD is $\begin{pmatrix}7\\11\\22\end{pmatrix}+\lambda\begin{pmatrix}18\\-2\\-32\end{pmatrix}$. Find E, the point of intersection of AC and BD. **3**

(d) Show that AC and BD are perpendicular **1**

(e) Show that E is the midpoint of AC. **1**

Question 15 continues on the following page

Question 15 (continued) **Marks**

(f) Given that E is also the midpoint of BD, show that $ABCD$ is a square. **1**

(g) Find the perpendicular distance from F to the line AC. **2**

(h) Find the equation of the line through E which is perpendicular to both AC and BD. **3**

(i) Show that F is on the line through E which is perpendicular to both AC and BD. **1**

(j) Find the volume of pyramid $ABCDF$. **1**

Examination continues on the following page

Question 16 ***(15 marks)*** **Marks**

(a) A rocket with mass m is launched vertically from the Earth's surface at a velocity of $U\ m/s$. It experiences resistance due to gravity of $\frac{mk}{x^2}$ where x is the distance from the *centre* of the Earth. If the Earth has a radius of R and letting the force due to gravity *at the Earth's surface* equal mg:

(i) Show that $k = gR^2$. **1**

(ii) Show that $v^2 = \frac{2gR^2}{x} + U^2 - 2gR$ and find the rocket's escape velocity (the speed at which it won't fall back to the Earth's surface) in terms of g and R. **2**

(iii) If the rocket is launched at a velocity of $U = \sqrt{\frac{3gR}{2}}$ find the maximum height it will reach in terms of g and R. **2**

(iv) Show that $v = \frac{\sqrt{4gR^2 - gRx}}{\sqrt{2x}}$ and find the time taken to reach the maximum height. **3**

Question 16 continues on the following page

Question 16 (continued) **Marks**

(b) (i) State the solutions to $z^7 - 1 = 0$. You may leave your answers in the form $cis\ \theta$. **1**

(ii) Hence show that **2**

$$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = \left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right)$$

(iii) By substituting $z = 1$ show that **2**

$$8\left(1 - \cos\frac{2\pi}{7}\right)\left(1 - \cos\frac{4\pi}{7}\right)\left(1 - \cos\frac{6\pi}{7}\right) = 7$$

(iv) Hence show that $\sin\frac{\pi}{7}\ \sin\frac{2\pi}{7}\ \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}$ **2**

$(Hint: \cos 2\theta = 1 - 2\sin^2\theta)$.

END OF EXAMINATION!!!!

GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2
2021 TRIAL HIGHER SCHOOL CERTIFICATE

Student Number: Solutions!

This Booklet contains the answer sheet for Section 1 and Writing Booklet for Section 2.

Section 1 ANSWER SHEET

Select the alternative A, B, C or D that best answers the question.

1.	A	●	B	○	C	○	D	○
2.	A	●	B	○	C	○	D	○
3.	A	○	B	○	C	●	D	○
4.	A	○	B	○	C	○	D	●
5.	A	○	B	●	C	○	D	○
6.	A	○	B	○	C	○	D	●
7.	A	○	B	●	C	○	D	○
8.	A	○	B	○	C	●	D	○
9.	A	○	B	○	C	●	D	○
10.	A	○	B	●	C	○	D	○

Instructions

- If you need more paper for Section 2, please ask your supervisor.
- Write your student number on every booklet you use.
- Write on both sides of each sheet of paper.

Total number of booklets used ______________.

Solutions:

(1) A (2) A (3) C (4) D (5) B (6) D (7) B (8) C (9) C (10) B

(1) $\sqrt{3^2+(-2)^2+1^2} = \sqrt{14}$ A

(2) $\underset{\sim}{r} = x = 2 + 3\lambda \Rightarrow \lambda = \frac{x-2}{3}$ (1)

$y = 1 - 4\lambda$ (2)

Sub. (1) in (2)

$y = 1 - 4\left(\frac{x-2}{3}\right)$

$3y = 3 - 4x + 8$ A

$4x + 3y - 11 = 0$

(3) By conjugate root theorem, other root $= 2 + 4i$.

By $\alpha + \beta = -\frac{b}{a}$

$(2-4i)+(2+4i) = -p$

$p = -4$. By $\alpha\beta = \frac{c}{a}$

$(2-4i)(2+4i) = q$

$20 = q$. C

$\therefore p = -4,\ q = 20$

(4) D

(5) $2e^{\frac{3\pi i}{4}}$ B

$= 2\text{cis}\frac{3\pi}{4}$

$= 2\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)$

$= -\sqrt{2} + i\sqrt{2}$

(6) Centre of motion $= 3$.

Amplitude $= 2$.

Period: $\frac{2\pi}{n} = \frac{\pi}{3} \Rightarrow n = 6$.

$x = 2\sin 6t + 3$. D

(7) B $3^3 - 1 = 26$ which is not divisible by 4.

(8) C "n is a positive integer BUT 3^n is NOT divisible by 4.

(9) $\int \frac{1}{x^2 - 4x + 7}\,dx$

$= \int \frac{1}{(x-2)^2 + 3}\,dx$ C

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + C$

(10) $\int x^{2021}\ln x\,dx$ $u = \ln x$ $v = \frac{1}{2022}x^{2022}$

$u' = \frac{1}{x}$ $v' = x^{2021}$

By $\int uv'\,dx = uv - \int vu'\,dx$

$\int x^{2021}\ln x\,dx$

$= \frac{x^{2022}}{2022}\ln x - \frac{1}{2022}\int x^{2021}\,dx$

$= \frac{x^{2022}}{2022}\ln x - \frac{1}{2022} \times \frac{x^{2022}}{2022} + C$ B

$= \frac{x^{2022}}{2022}\left[\ln x - \frac{1}{2022}\right] + C$

Q(11)(a)(i) $\frac{z}{w}$

$= \frac{1+2i}{3+i} \times \frac{(3-i)}{(3-i)}$

$= \frac{5+5i}{10}$

$= \frac{1+i}{2}$.

(ii) $\text{Arg } z = \tan^{-1}(2)$

$\text{Arg } w = \tan^{-1}\left(\frac{1}{3}\right)$

$\text{Arg}\left(\frac{z}{w}\right) = \tan^{-1}(1)$

$= \frac{\pi}{4}$.

As $\text{Arg}\left(\frac{z}{w}\right) = \text{Arg } z - \text{Arg } w$,

$\tan^{-1}(2) - \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$.

(b) $(\sqrt{3} - i)^5$

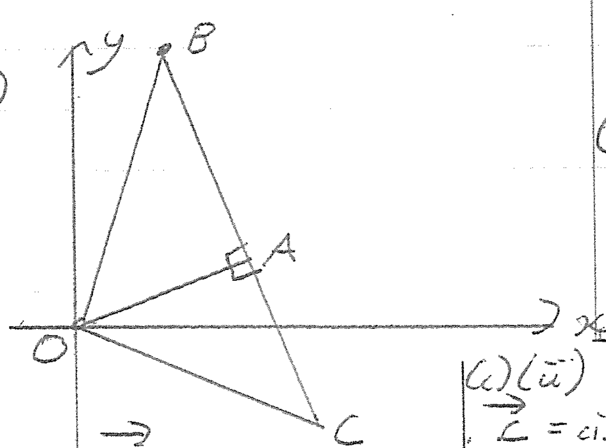
$= \left[2\text{cis}\left(-\frac{\pi}{6}\right)\right]^5$

$= 32\,\text{cis}\left(-\frac{5\pi}{6}\right)$

$= 32\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

$= -16\sqrt{3} - 16i$.

(c) (i)



$\overrightarrow{OA} = z_1$

$\overrightarrow{AB} = \sqrt{3}\,\text{cis}\,\frac{\pi}{2} \times z_1$

$= i\sqrt{3}\, z_1$.

$z_2 = \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= z_1 + i\sqrt{3}\, z_1$

$= z_1(1 + i\sqrt{3})$.

(c)(ii)

$\overrightarrow{AC} = \text{cis}\left(-\frac{\pi}{2}\right) \times \sqrt{3} \times z_1$

$= -i\sqrt{3}\, z_1$.

$\therefore z_3 = \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

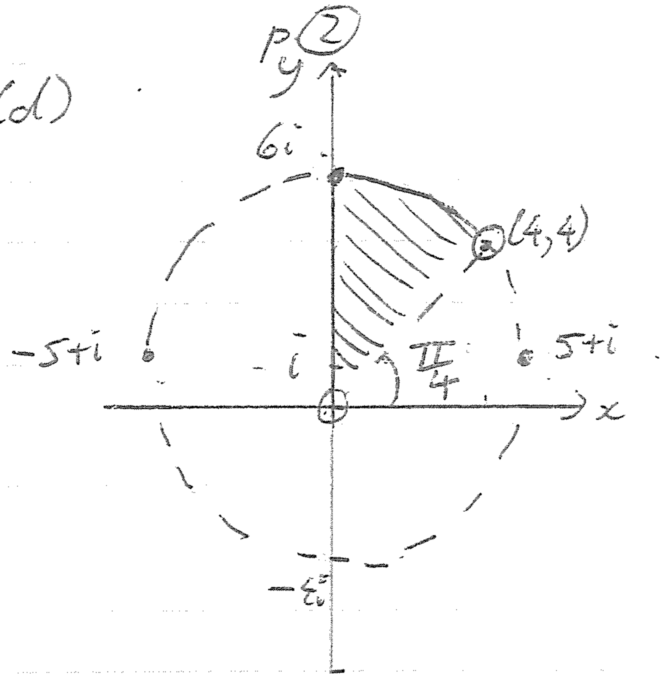
$= z_1 - i\sqrt{3}\, z_1$

$= (1 - i\sqrt{3}) z_1$.

$\therefore z_2 + z_3 = (1 + i\sqrt{3}) z_1 + (1 - i\sqrt{3}) z_1$

$= 2z_1$.

(d)



(e)(i) By Euler, $e^{ni\theta} = \cos n\theta + i\sin n\theta$

$e^{-ni\theta} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$ [as cos even sin odd].

$\therefore e^{ni\theta} - e^{-ni\theta}$

$= (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$.

(ii) Hence if $z = e^{i\theta}$

$\left(z - \frac{1}{z}\right)^4 = z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$

$= \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$(2i\sin\theta)^4 = 2\cos 4\theta - 8\cos 2\theta + 6$

$16\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$

(iii) $\int \sin^4\theta \, d\theta = \int \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + 6 \, d\theta$

$= \frac{1}{32}\sin 4\theta - \frac{1}{4}\sin 2\theta + 6\theta + C$.

Q.1(2)(a)(i) $\dfrac{9x-9}{(x-2)^2(x+1)} = \dfrac{A}{(x-2)^2} + \dfrac{B}{(x-2)} + \dfrac{C}{(x+1)}$

$9x - 9 = A(x+1) + B(x-2)(x+1) + C(x-2)^2 \quad (1)$

Sub. $x = -1$ in (1):

$-18 = C \times (-3)^2$

$-18 = 9C$

$\underline{-2 = C.}$

Sub. $x = 2$ in (1):

$9 = A(2+1)$

$\underline{3 = A.}$

Sub. $A = 3$, $C = -2$ & $x = 0$ in (1):

$-9 = 3 - 2B - 8$

$-9 = -2B - 5.$

$\underline{2 = B.}$

$\therefore \dfrac{9x-9}{(x-2)^2(x+1)} = \dfrac{3}{(x-2)^2} + \dfrac{2}{(x-2)} - \dfrac{2}{(x+1)}$

(ii) Hence $\displaystyle\int \frac{9x-9}{(x-2)^2(x+1)} \, dx$

$= \displaystyle\int \frac{3}{(x-2)^2} + \frac{2}{(x-2)} - \frac{2}{(x+1)} \, dx$

$= \dfrac{-3}{(x-2)} + 2\ln\left(\dfrac{x-2}{x+1}\right) + C.$

(b) $\displaystyle\int \frac{1}{\sin x + \cos x + 1} \, dx$

$t = \tan\left(\frac{x}{2}\right)$

$\dfrac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \dfrac{1+t^2}{2}$

$\therefore dx = \dfrac{dx}{dt} \, dt = \dfrac{2}{1+t^2} \, dt$

$= \displaystyle\int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1} \cdot \frac{2}{1+t^2} \, dt$

$= \displaystyle\int \frac{2}{2t + 1 - t^2 + 1 + t^2} \, dt$

$= \displaystyle\int \frac{1}{t+1} \, dt$

$= \ln(t+1) + C = \underline{\ln\left[\tan^{-1}\left(\frac{x}{2}\right) + 1\right] + C.}$

Q.(12)(i) $\int_0^{\pi} x\sin x \, .dx$ $u = x$ $v = -\cos x$
$u' = 1$ $v' = \sin x$

By $\int u v' . dx = uv - \int v u' . dx$

$$\int_0^{\pi} x\sin x . dx = \left[-x\cos x\right]_0^{\pi} + \int_0^{\pi} \cos x . dx$$
$$= \left[-\pi\cos\pi\right] + \left[\sin x\right]_0^{\pi}$$
$$= \pi + 0$$
$$= \underline{\pi.}$$

(d) (i) $\int_0^a f(a-x).dx$ Let $u = a - x$
$du = 1.dx$

$$= -\int_0^a f(a-x). -1 .dx$$
$$= -\int_{u=a}^{u=0} f(u).du$$
$$= \int_{u=0}^{u=a} f(u).du \quad \left(\text{as } \int_a^b f(x).dx = -\int_b^a f(x).dx\right].$$
$$= \int_0^a f(x).dx \text{ [using } u \text{ as a "dummy" variable].}$$

(ii) Hence $\int_0^{\pi} x\sin x .dx = \int_0^{\pi} (\pi - x)\sin(\pi - x).dx$
$$= \int_0^{\pi} (\pi - x)\sin x \; .dx \quad [\text{as } \sin(\pi - x) = \sin x].$$
$$= \pi\int_0^{\pi} \sin x .dx - \int_0^{\pi} x\sin x .dx \qquad + \int_0^{\pi} x\sin x .dx$$

$$2\int_0^{\pi} x\sin x .dx = \pi\int_0^{\pi} \sin x .dx$$
$$= \pi\left[-\cos x\right]_0^{\pi}.$$
$$2\int_0^{\pi} x\sin x .dx = 2\pi .$$
$$\underline{\int_0^{\pi} x\sin x \, dx = \pi .}$$

Q.(12)(e)(i) $I_n = \int \sin^n x . dx$

$$= \int \sin^{n-1} x . \sin x . dx \qquad u = \sin^{n-1} x \quad v = -\cos x$$
$$u' = (n-1)\sin^{n-2} x \cos x \quad v' = \sin x$$

By $\int u \, v' . dx = uv - \int v u' . dx$

$$\int \sin^n x . dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x . dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x (1 - \sin^2 x) . dx$$

$$\int \sin^n x . dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x . dx - (n-1)\int \sin^n x . dx$$

$$\therefore I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$n I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}.$$

$$\therefore I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{(n-1)}{n} I_{n-2}.$$

(ii) $I_0 = \int 1 . dx = x.$

$$I_2 = -\frac{\cos x \sin x}{2} + \frac{1}{2} I_0$$

$$= -\frac{\cos x \sin x}{2} + \frac{1}{2} x.$$

$$I_4 = -\frac{\cos x \sin^3 x}{4} + \frac{3}{4} I_2$$

$$= -\frac{\cos x \sin^3 x}{4} + \frac{3}{4}\left[-\frac{\cos x \sin x}{2} + \frac{1}{2} x\right]$$

$$\int \sin^4 x . dx = -\frac{\cos x \sin^3 x}{4} - \frac{3 \cos x \sin x}{8} + \frac{3}{8} x + C.$$

Q. (13) (a) If a is even,
a is either divisible by 4 i.e. $a = 4k$
or a is not i.e. $a = 4k+2$.

Case 1: $a = 4k$.

$$a^2 + 2a$$
$$= a(a+2)$$
$$= 4k(4k+2)$$
$$= 8k(2k+1)$$

which is divisible by 8.

Case 2: $a = 4k+2$

$$a^2+2a$$
$$= a(a+2)$$
$$= (4k+2)(4k+4)$$
$$= 2(2k+1) \times 4(k+1)$$
$$= 8(2k+1)(k+1)$$

which is divisible by 8.

(b) Let $\sqrt{5}$ be rational
i.e. $\sqrt{5} = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 1$.
& p, q have NO common factors.

$\therefore$ Squaring BS:
$$5 = \frac{p^2}{q^2}$$
$$5q^2 = p^2 \quad (1)$$
$\therefore$ 5 is a factor of p^2 (as q^2 is NOT)
5 is a factor of p
$$\therefore p = 5k, k \in \mathbb{Z}$$
$$p^2 = 25k^2 \quad (2)$$
Sub. $p^2 = 25k^2$ (2) in (1)
$$5q^2 = 25k^2$$
$$q^2 = 5k^2$$
$\therefore$ 5 is a factor of q^2
$\Rightarrow$ 5 is a factor of q.
<u>BUT</u> p & q have NO COMMON FACTORS.
$\therefore$ There must be a contradiction
$\sqrt{5}$ is IRRATIONAL.

(c) Contrapositive of if $n^2 - 2n$ odd n odd is:
If n is EVEN, $n^2 - 2n$ is EVEN.
Let $n = 2k$.
$$n^2 - 2n = (2k)^2 - 2 \times 2k.$$
$$= 4k^2 - 2k$$
$$= 2(2k^2 - k)$$
which is even.

Q. (13) (d) Show true for $n=1$

LHS $= \frac{1}{1^2} = 1.$ RHS $= \frac{3}{2} - \frac{1}{1+1} = 1.$

LHS $\geq$ RHS

True for $n=1$.

Assume true for $n=k$:

i.e. $\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} \geq \frac{3}{2} - \frac{1}{k+1}.$

Prove true for $n=k+1$

i.e. RTP $\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \geq \frac{3}{2} - \frac{1}{k+2}.$

LHS

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2}$$

$$\geq \frac{3}{2} - \frac{1}{k+1} + \frac{1}{(k+1)^2} \text{ (By assumption).}$$

$$= \frac{3}{2} - \left[\frac{1}{(k+1)} - \frac{1}{(k+1)^2}\right]$$

$$= \frac{3}{2} - \left[\frac{k+1-1}{(k+1)^2}\right]$$

$$= \frac{3}{2} - \frac{k}{(k+1)^2}.$$

$$= \frac{3}{2} - \frac{k(k+2)}{(k+1)^2(k+2)}$$

$$= \frac{3}{2} - \frac{(k+1)^2 - 1}{(k+1)^2(k+2)}$$

$$= \frac{3}{2} - \frac{(k+1)^2}{(k+1)^2(k+2)} + \frac{1}{(k+1)^2(k+2)}$$

$$= \frac{3}{2} - \frac{1}{k+2} + \frac{1}{(k+1)^2(k+2)}$$

$$> \frac{3}{2} - \frac{1}{k+2}, \ k \geq 1.$$

= RHS QED.

If it is true for $n=k$ it will be true for $n=k+1$. Hence as it is true for $n=1$ it will be true for $n=1+1=2$ & so on $\forall\ n \in Z^+$.

(e) (i) If $a, b \in R$

$a-b \in R$

$\therefore (a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab.$

(ii) If $a > 0$

$\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0$

$a + \frac{1}{a} - 2 \geq 0$

$a + \frac{1}{a} \geq 2.$

Hence $\left(a + \frac{1}{b}\right)\left(\frac{1}{a} + \frac{1}{b}\right)$

$= 1 + \left(\frac{a}{b} + \frac{b}{a}\right) + 1$ $\left[\text{as } \frac{b}{a} = \frac{1}{\left(\frac{a}{b}\right)}\right].$

$\geq 1 + 2 + 1$

$= 4$

(iii) Hence $(\sin^2\theta + \cos^2\theta)(\operatorname{cosec}^2\theta + \sec^2\theta) \geq$

$\operatorname{cosec}^2\theta + \sec^2\theta \geq 4$

(as $\sin^2\theta + \cos^2\theta = 1$).

Q.(14)(a)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x$

$\frac{d}{dx}(v^2) = -18x$

$v^2 = \int -18x \, dx$

$v^2 = -9x^2 + C$

As $v = 9$ when $x = 4$,

$9^2 = -9 \times 4^2 + C$

$225 = C$

$v^2 = 225 - 9x^2$

$v^2 = 9(25 - x^2)$

$n^2 = 9 \Rightarrow n = 3$ Period $= \frac{2\pi}{3}$. $a = 25 \Rightarrow a = 5$. Amplitude $= 5$.

NOTE: As $v^2 = n^2(a^2 - x^2)$ & $\ddot{x} = -n^2 x$ are ON formula sheet. Students CAN do $n^2 = 9 \Rightarrow n = 3$.

$\therefore v^2 = 3^2(a^2 - x^2)$

$v = 9$ when $x = 4$

$9^2 = 9(a^2 - 4^2)$

$9 = a^2 - 16$

$25 = a^2$

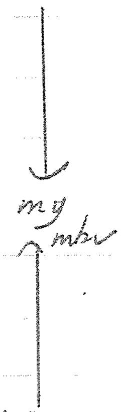
$\underline{5 = a}$.

(ii) Maximum velocity. Occurs where $\ddot{x} = -9x = 0 \Rightarrow x = 0$.

$v^2 = 9(25 - 0)$

$v = 25$ m/s maximum.

(b)(i)



$F = ma = mg - mkv$.

$a = g - kv$.

$\underline{a = 10 - \frac{v}{6}}$.

Terminal velocity is when $a = 0$

$\underline{\text{i.e. } 10 - \frac{v}{6} = 0 \Rightarrow v = 60\text{m/s}}$.

(ii) $\frac{dv}{dt} = 10 - \frac{v}{6}$

$= \frac{60 - v}{6}$

$\therefore \frac{dt}{dv} = \frac{6}{60 - v}$.

$t = -6 \int \frac{-1}{60 - v} \, dv$

$= -6 \ln(60 - v) + C$.

As $v = 0$ when $t = 0$

$0 = -6 \ln 60 + C$

$6 \ln 60 = C$.

$t = -6 \ln(60 - v) + 6 \ln 60$

$= -6 \ln\left(\frac{60 - v}{60}\right)$

$t = -6 \ln\left(1 - \frac{v}{60}\right)$.

Hits ground at half terminal v.

$= 30$m/s.

$t = -6 \ln\left(1 - \frac{30}{60}\right)$

$= 6 \ln 2$ seconds ($\doteqdot 4.1588..$s)

Q.(14)(b)(iii) $t = -6\ln\left(1 - \frac{v}{60}\right)$

$$e^{-\frac{t}{6}} = 1 - \frac{v}{60}$$

$$v = 60 - 60e^{-\frac{t}{6}}$$

$$x = \int 60 - 60e^{-\frac{t}{6}}\,dt$$

$$x = 60t + 360e^{-\frac{t}{6}} + C$$

As $x = 0$ when $t = 0$

$$0 = 60 \times 0 + 360e^{0} + C$$

$$-360 = C$$

$$x = 60t + 360e^{-\frac{t}{6}} - 360$$

$$x = 60t + 360\left(e^{-\frac{t}{6}} - 1\right)$$

Particle lands on ground after $6\ln 2$ s.

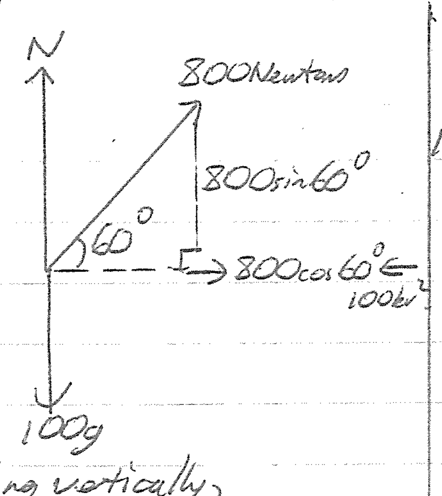
$$x = 60 \times 6\ln 2 + 360\left[e^{-\ln 2} - 1\right].$$

$$= (360\ln 2 - 180)\text{m}$$

$$\doteqdot \underline{69.53..\text{m}.}$$

The particle was dropped from a height of 69.53.. m.

(c)(i)



Resolving vertically,

$$N + 800\sin 60^{\circ} = 100g$$

$$N = (1000 - 400\sqrt{3})\text{Newtons}$$

$$\doteqdot \underline{307.2\text{Newtons}.}$$

(ii) Resolving horizontally,

$$F = 800\cos 60^{\circ} - 100kv^2.$$

$$100a = 400 - 100kv^2.$$

$$a = 4 - kv^2.$$

Limiting horizontal velocity:

$$a = 0$$

$$4 = kv^2.$$

$$v = \sqrt{\frac{4}{k}}.$$

If limiting $v = 20$

$$\sqrt{\frac{4}{k}} = 20$$

$$\frac{4}{k} = 400$$

$$\underline{k = \frac{1}{100}.}$$

Q.(14)(c)(iii) $\ddot{x} = v.\frac{dv}{dx} = 4 - kv^2$.

$$\frac{dv}{dx} = \frac{4-kv^2}{v}.$$

$$\frac{dx}{dv} = \frac{v}{4-kv^2}$$

$$x = \int \frac{v}{4-kv^2} . dv$$

$$= -\frac{1}{2k} \int \frac{2kv}{4-kv^2} . dv$$

$$x = -\frac{1}{2k} \ln(4-kv^2) + C$$

As $x = 0$ when $v = 0$,

$$0 = -\frac{1}{2k} \ln 4 + C.$$

$$\frac{1}{2k} \ln 4 = C.$$

$$x = -\frac{1}{2k} \ln\left(\frac{4-kv^2}{4}\right)$$

$$x = -\frac{1}{2k} \ln\left(1 - \frac{kv^2}{4}\right).$$

When $v = 10$, $x = -\frac{1}{2 \times \frac{1}{100}} \ln\left(1 - \frac{10^2 \times \frac{1}{100}}{4}\right)$

$$= 50 \ln\left(\frac{4}{3}\right) m$$

$$\underline{x = 14.38.. m.}$$

Q.(15)(a) $4\times 1 - 12\times 3 + 3\times -2 + 38 = 0$

A lies on the plane.

(b) Direction vector $\underset{\sim}{AC} = \begin{pmatrix} 31-1 \\ 17-3 \\ 14+2 \end{pmatrix} = \begin{pmatrix} 30 \\ 14 \\ 16 \end{pmatrix}$

As AC passes through A, $AC = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 30 \\ 14 \\ 16 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 30 \\ 14 \\ 16 \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \\ 22 \end{pmatrix} + \lambda_2 \begin{pmatrix} 18 \\ -2 \\ -32 \end{pmatrix}$

$1 + 30\lambda_1 = 7 + 18\lambda_2 \ (1) \Rightarrow 30\lambda_1 - 18\lambda_2 = 6 \Rightarrow 5\lambda_1 - 3\lambda_2 = 1 \ (1)$

$3 + 14\lambda_1 = 11 - 2\lambda_2 \ (2) \Rightarrow 14\lambda_1 + 2\lambda_2 = 8 \Rightarrow 7\lambda_1 + \lambda_2 = 4 \ (2) \times 3 = (4)$

$-2 + 16\lambda_1 = 22 - 32\lambda_2 \ (3) \Rightarrow 16\lambda_1 + 32\lambda_2 = 24 \Rightarrow 2\lambda_1 + 4\lambda_2 = 3 \ (3)$ (1)+(4)

$5\lambda_1 - 3\lambda_2 = 1 \ (1)$
$21\lambda_1 + 3\lambda_2 = 12 \ (4)$ +

$26\lambda_1 = 13$
$\lambda_1 = \frac{1}{2}$.

If $\lambda_1 = \frac{1}{2}$
Sub. in (2):
$7 \times \frac{1}{2} + \lambda_2 = 4$.
$\lambda_2 = \frac{1}{2}$.

Note: Even if you use equations (1) & (3) or (2) & (3) initially, $\lambda_1 = \lambda_2 = \frac{1}{2}$.

Sub. $\lambda_1 = \lambda_2 = \frac{1}{2}$ in (3): $2 \times \frac{1}{2} + 4 \times \frac{1}{2} = 3$.

Point of intersection $= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 30 \\ 14 \\ 16 \end{pmatrix} = \begin{pmatrix} 16 \\ 10 \\ 6 \end{pmatrix} = E$ [or (16,10,6)].

(d) Direction vector AC . Direction vector BD

$= \begin{pmatrix} 30 \\ 14 \\ 16 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ -2 \\ -32 \end{pmatrix}$

$= 0$.

$\therefore AC \perp BD$.

(e) Midpoint $AC = \left(\frac{1+31}{2}, \frac{3+17}{2}, \frac{-2+14}{2}\right) = (16, 10, 6) = E$.

PTO →

(f) AC & BD bisect each other at right-angles

$\therefore$ ABCD is a rhombus.

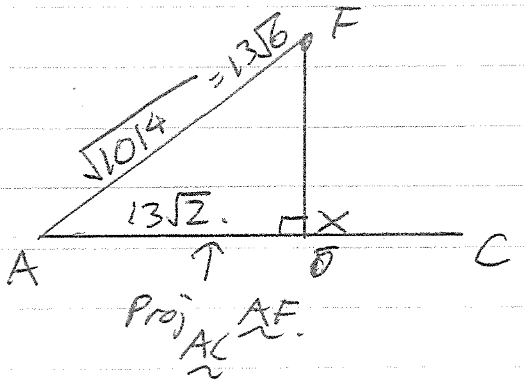
$$\text{Direction } AB \cdot \text{Direction } BC = \begin{pmatrix}6\\8\\24\end{pmatrix}\cdot\begin{pmatrix}24\\6\\-8\end{pmatrix} = 0.$$

$AB \perp BC$

$\therefore$ ABCD is a rhombus with a right $\angle$.

ABCD is a square.

(g) $F = (24, -14, 12)$



$$\left|Proj_{\underset{\sim}{AC}} \underset{\sim}{AF}\right| = \frac{\underset{\sim}{AC}\cdot\underset{\sim}{AF}}{|\underset{\sim}{AC}|}.$$

$$= \frac{\begin{pmatrix}30\\14\\16\end{pmatrix}\cdot\begin{pmatrix}23\\-17\\14\end{pmatrix}}{\sqrt{30^2+14^2+16^2}}$$

$$= \frac{676}{\sqrt{1352}}$$

$$= 13\sqrt{2}.$$

$|\underset{\sim}{AF}|$

$= \sqrt{23^2+17^2+14^2}$

$= \sqrt{1014}$

$$FX = \sqrt{(13\sqrt{6})^2-(13\sqrt{2})^2} = \underline{26.}$$

(h) $E = \begin{pmatrix}16\\10\\6\end{pmatrix}$ $\perp$ to $\begin{pmatrix}30\\14\\16\end{pmatrix}$ & $\begin{pmatrix}18\\-2\\-32\end{pmatrix}$

So direction vector $= \begin{pmatrix}x\\y\\z\end{pmatrix}$

$30x + 14y + 16z = 0 \ (1)$

$18x - 2y - 32z = 0$

$9x - y - 16z = 0 \ (2)$

$$\begin{array}{rl} 30x+14y+16z=0 & (1) \\ 9x-y-16z=0 & (2)\ + \\ \hline 39x+13y \quad =0 & \\ \underline{3x+y \quad =0} & .\ y=-3x. \end{array}$$

Sub. $y = -3x$ in (2): $9x+3x-16z=0$

$12x - 16z = 0$

$\underline{3x = 4z.}$

Let $y = 12.$

$x = -4.$

$3\times -4 = 4z \Rightarrow z = -3.$

Direction vector $= \underline{\begin{pmatrix}-4\\12\\-3\end{pmatrix}}.$

Line is $\underline{\begin{pmatrix}16\\10\\6\end{pmatrix} + \lambda\begin{pmatrix}-4\\12\\-3\end{pmatrix}}.$

Q.1(15)(i) Showing $F\begin{pmatrix}24\\-14\\12\end{pmatrix}$ is on $\begin{pmatrix}16\\10\\6\end{pmatrix}+\lambda\begin{pmatrix}-4\\12\\-3\end{pmatrix}$

x:	y:	z:
$16-4\lambda=24$	$10+12\lambda=-14$	$6-3\lambda=12$
$\lambda=-2.$	$\lambda=-2.$	$\lambda=-2.$

As $\lambda=-2$ in all cases, F is on $\perp$ line.

(j) This means EF is the height of the pyramid.

As ABCD is a square.

$AB = \sqrt{6^2+8^2+24^2} = \sqrt{676}.$

$\therefore V = \frac{1}{3}Ah$

$= \frac{1}{3}\times(\sqrt{676})^2\times 26$

$= 5858\frac{2}{3}u^3.$

Q.(16)(a)(i)

$$\downarrow \frac{mk}{x^2}.$$

Note: $\frac{mk}{x^2} = mg$ when $x = R$.

$$\frac{mk}{R^2} = mg \quad \times \frac{R^2}{m}.$$

$$k = gR^2.$$

(ii) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{-gR^2}{x^2}.$

$$\frac{d}{dx}\left(v^2\right) = \frac{-2gR^2}{x^2}.$$

$$v^2 = \int \frac{-2gR^2}{x^2}.dx$$

$$v^2 = \frac{2gR^2}{x} + C$$

As $v = U$ when $x = R$,

$$U^2 = \frac{2gR^2}{R} + C$$

$$U^2 - 2gR = C.$$

$$v^2 = \frac{2gR^2}{x} + U^2 - 2gR.$$

As $\frac{2gR^2}{x} > 0$ [as $x > 0$], if $U^2 - 2gR > 0$ then $v^2 > 0$ & rocket will never stop.

$$\therefore \quad U^2 > 2gR.$$

$$U > \sqrt{2gR}.$$

Escape velocity $= \sqrt{2gR}.$

(iii) $v^2 = \frac{2gR^2}{x} + \frac{3gR}{2} - 2gR$

$$= \frac{2gR^2}{x} - \frac{gR}{2}.$$

Max. height is x when $v = 0$

$$\frac{2gR^2}{x} = \frac{gR}{2}$$

$$4gR^2 = gRx. \quad (\times 2x)$$

$$\underline{4R = x.}$$

Max. height reached is $4R$ above CENTRE of Earth [$3R$ above surface].

Q.(16)(a)(iv) $v^2 = \frac{2gR^2}{x} - \frac{gR}{2}$

$= \frac{4gR^2 - gRx}{2x}$

∴ v (positive as rocket ascends)

$= \sqrt{\frac{4gR^2 - gRx}{2x}}$

$v = \frac{\sqrt{4gR^2 - gRx}}{\sqrt{2x}}$

$\frac{dx}{dt} = \frac{\sqrt{4gR^2 - gRx}}{\sqrt{2x}}$

$\frac{dt}{dx} = \frac{\sqrt{2x}}{\sqrt{4gR^2 - gRx}}$

$t = \int_R^{4R} \frac{\sqrt{2x}}{\sqrt{4gR^2 - gRx}} \, dx$

$= \int_R^{4R} \frac{2x}{\sqrt{8gR^2x - 2gRx^2}} \, dx$

$= \int_R^{4R} \frac{2x}{\sqrt{2gR}\sqrt{4Rx - x^2}} \, dx$

$= -\frac{1}{\sqrt{2gR}} \int_R^{4R} \frac{-2x}{\sqrt{4Rx - x^2}} \, dx$

$= -\frac{1}{\sqrt{2gR}} \int_R^{4R} \frac{4R - 2x}{\sqrt{4Rx - x^2}} - \frac{4R}{\sqrt{4R^2 - (x - 2R)^2}} \, dx$

$= -\frac{1}{\sqrt{2gR}} \left[2\sqrt{4Rx - x^2} - 4R\sin^{-1}\left(\frac{x - 2R}{2R}\right) \right]_R^{4R}$

$t = \sqrt{\frac{R}{g}} \left[\frac{4\pi\sqrt{2}}{3} + \sqrt{6} \right]$ seconds.

(16)(b)(i) Solutions $= cis\frac{2\pi}{7}, cis\frac{4\pi}{7}, cis\frac{6\pi}{7},$
$cis\frac{12\pi}{7}$ $cis\frac{10\pi}{7}$ $cis\frac{8\pi}{7}$ & 1.

(ii) As $z^7 - 1 = 0$

is $(z-1)(z^6+z^5+z^4+z^3+z^2+z+1) = 0$

$cis\frac{2\pi}{7} \ldots cis\frac{12\pi}{7}$ are solutions to $z^6+z^5+z^4+z^3+z^2+z+1=0$

$\therefore z^6+z^5+z^4+z^3+z^2+z+1 = \left(z-cis\frac{2\pi}{7}\right)\left(z-cis\frac{12\pi}{7}\right)\left(z-cis\frac{4\pi}{7}\right)\left(z-cis\frac{10\pi}{7}\right)$
$\left(z-\frac{6\pi}{7}\right)\left(z-cis\frac{8\pi}{7}\right).$

Taking $\left(z-cis\frac{2\pi}{7}\right)\left(z-cis\frac{12\pi}{7}\right)$

$= \left(z^2 - z\left(cis\frac{2\pi}{7}+cis\frac{12\pi}{7}\right) + cis\frac{2\pi}{7}\times cis\frac{12\pi}{7}\right)$

$= \left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)$

Cos $cis\frac{2\pi}{7}+cis\frac{12\pi}{7}$
$=\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)+$
$\left(\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}\right)$
$=2\cos\frac{2\pi}{7}.$
& $cis\frac{2\pi}{7}\times cis\frac{12\pi}{7} = cis\,2\pi = 1.$

Similarly

$\left(z-cis\frac{4\pi}{7}\right)\left(z-cis\frac{10\pi}{7}\right)$

$= z^2 - 2z\cos\frac{4\pi}{7} + 1$

& $\left(z-cis\frac{6\pi}{7}\right)\left(z-cis\frac{8\pi}{7}\right) = z^2 - 2z\cos\frac{6\pi}{7} + 1.$

$\therefore z^6+z^5+z^4+z^3+z^2+z+1 = \left(z^2-2z\cos\frac{2\pi}{7}+1\right)\left(z^2-2z\cos\frac{4\pi}{7}+1\right)\left(z^2-2z\cos\frac{6\pi}{7}+1\right)$

(iii) Substituting $z=1$ in (ii)

$1^6+1^5+1^4+1^3+1^2+1+1 = \left(1^2-2\cos\frac{2\pi}{7}+1\right)\left(1^2-2\cos\frac{4\pi}{7}+1\right)\left(1^2-2\cos\frac{6\pi}{7}+1\right)$

$7 = \left(2-2\cos\frac{2\pi}{7}\right)\left(2-2\cos\frac{4\pi}{7}\right)\left(2-2\cos\frac{6\pi}{7}\right)$

$7 = 2\times2\times2\left(1-\cos\frac{2\pi}{7}\right)\left(1-\cos\frac{4\pi}{7}\right)\left(1-\cos\frac{6\pi}{7}\right)$

$7 = 8\left(1-\cos\frac{2\pi}{7}\right)\left(1-\cos\frac{4\pi}{7}\right)\left(1-\cos\frac{6\pi}{7}\right).$

(iv) As $\cos 2\theta = 1-2\sin^2\theta$

$1-\cos\frac{2\pi}{7}$
$= 1-\left(1-2\sin^2\frac{\pi}{7}\right)$
$= 2\sin^2\frac{\pi}{7}.$

Similarly, $1-\cos\frac{4\pi}{7} = 2\sin^2\frac{2\pi}{7}$

& $1-\cos\frac{6\pi}{7} = 2\sin^2\frac{3\pi}{7}.$

$\therefore\ 7 = 8\times2\sin^2\frac{\pi}{7}\times2\sin^2\frac{2\pi}{7}\times2\sin^2\frac{3\pi}{7}.$

$7 = 64\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin^2\frac{3\pi}{7}.$ Taking $\sqrt{\ }$ of BS & noting $\cos\frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}$ in Q, so all sines positive.

$\sqrt{7} = 8\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} \Rightarrow \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}$

END OF SOLUTIONS!